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The biHermitian topological sigma model

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Abstract: BiHermitian geometry, discovered long ago by Gates, Hull and Roček, is the most general sigma model target space geometry allowing for $(2, 2)$ world sheet supersymmetry. By using the twisting procedure proposed by Kapustin and Li, we work out the type A and B topological sigma models for a general biHermtian target space, we write down the explicit expression of the sigma model's action and BRST transformations and present a computation of the topological gauge fermion and the topological action.

Keywords: Topological Field Theories, Sigma Models, Differential and Algebraic Geometry, BRST Symmetry.

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1. Introduction

Type II superstring Calabi-Yau compactifications are described by $(2,2)$ superconformal sigma models with Calabi-Yau target manifolds. These field theories are however rather complicated and, so, they are difficult to study. In 1988, Witten showed that a $(2,2)$ supersymmetric sigma model on a Calabi-Yau space could be twisted in two different ways, to give the so called A and B topological sigma models [1, 2]. Unlike the original untwisted sigma model, the topological models are soluble: the calculation of observables can be reduced to standard problems of geometry and topology. For the A model, the ring of observables is found to be a deformation of the complex de Rham cohomology $\bigoplus_p H^p(M,\mathbb{C})_{\rm qu}$ going under the name of quantum cohomology. For the B model, the ring of observables turns out to be isomorphic to $\bigoplus_{p,q} H^p(\wedge^q T^{1,0}M)$. Furthermore, all correlators of the A model are symplectic invariants of M, while all correlators of the B model are invariants of the complex structure on M. For this reason, the topological sigma models constitute an ideal and convenient field theoretic ground for the study of 2-dimensional supersymmetric field theories.

Witten's analysis was restricted to the case where the sigma model target space geometry was Kaehler. In a classic paper, Gates, Hull and Roček [3] showed that, for a

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2-dimensional sigma model, the most general target space geometry allowing for $(2,2)$ supersymmetry was biHermitian or Kaehler with torsion geometry. This is characterized by a Riemannian metric g_{ab}, two generally non commuting complex structures $K_\pm{}^a{}_b$ and a closed 3-form H_{abc}, such that g_{ab} is Hermitian with respect to both the $K_\pm{}^a{}_b$ and the $K_\pm{}^a{}_b$ are parallel with respect to two different metric connections with torsion proportional to $\pm H_{abc}$ [4 − 7]. This geometry is more general than that considered by Witten, which corresponds to the case where $K_+{}^a{}_b = \pm K_-{}^a{}_b$ and $H_{abc} = 0$. So, the natural question arises as to construct topological sigma models with biHermitian target space.

A turning point in the quest towards accomplishing this goal was the realization that biHermitian geometry is naturally expressed in the language of generalized complex and Kaehler geometry worked out by Hitchin and Gualtieri [8 − 10]. Many attempts have been made to construct sigma models with generalized complex or Kaehler target manifolds, by invoking world sheet supersymmetry, employing the Batalin-Vilkovisky quantization algorithm, etc. [12 − 25]. All these were somehow unsatisfactory either because they remained confined to the analysis of geometrical aspects of the sigma models or because they yielded field theories, which though interesting in their own, were not directly suitable for quantization, showed no apparent kinship with Witten's A and B models and were of limited relevance for string theory.

In their seminal paper [13], Kapustin and Li defined and studied the analogues of the A and B models for the general biHermitian $(2,2)$ supersymmetric sigma model. They tackled several crucial issues.

(a) They formulated their analysis in the natural framework of generalized complex and Kaehler geometry.

(b) They identified the appropriate twisting prescriptions yielding the biHermitian A and B models.

(c) They showed that the consistency of the quantum theory requires one of the two twisted generalized complex structures forming the target space twisted generalized Kaehler structure to be a twisted generalized Calabi-Yau structure.

(d) They showed that the BRST cohomology is isomorphic to the cohomology of the Lie algebroid associated with that structure.

However, Kapustin and Li left much work to be done.

(e) They did not write down the explicit expression of the action S_t of the biHermitian A and B models.

(f) They provided only partial expressions of the BRST symmetry operator s_t.

(g) They left unsolved the problem of writing the action in the form

$$S_t = s_t\Psi_t + S_{\text{top}}, \tag{1.1}$$

where Ψ_t is a ghost number -1 gauge fermion and S_{top} is a topological action, as required by the topological nature of the model.

In this paper, we have carried out these missing calculations and written down all the required expressions. It is our belief that the completeness of the theory definitely demands this work to be done. There still are open problems with point g above. Their solution is left for future work.

The paper is organized as follows. In section 2, we review the basic notions of biHermitian and generalized complex and Kaehler geometry used in the paper. In section 3, we review the main properties of the biHermitian (2,2) supersymmetric sigma model, which are relevant in the following analysis. In section 4, we implement the twisting prescriptions of Kapustin and Li and write down the explicit expressions of the action S_t and of the BRST symmetry operator s_t of the biHermitian A and B models. In section 5, we study the ghost number anomaly and the descent formalism. In section 6, we compute the gauge fermion Ψ_t and the topological action S_{top} appearing in (1.1). Finally, in the appendices, we conveniently collect the technical details of our analysis.

After this work was completed, we became aware of the paper [26], where similar results were obtained.

2. BiHermitian geometry

The target space geometry of the sigma models studied in the following is biHermitian. Below, we review the basic facts of biHermitian geometry and its relation to generalized Kaehler geometry.

Let M be a smooth manifold. A biHermitian structure $(g, H, K_\pm)$ on M consists of the following elements.

(a) A Riemannian metric g_{ab}.[1]

(b) A closed 3-form H_{abc}

$$\partial_{[a}H_{bcd]} = 0. \tag{2.1}$$

(c) Two complex structures $K_\pm{}^a{}_b$,

$$K_\pm{}^a{}_c K_\pm{}^c{}_b = -\delta^a{}_b, \tag{2.2}$$

$$K_\pm{}^d{}_a \partial_d K_\pm{}^c{}_b - K_\pm{}^d{}_b \partial_d K_\pm{}^c{}_a - K_\pm{}^c{}_d \partial_a K_\pm{}^d{}_b + K_\pm{}^c{}_d \partial_b K_\pm{}^d{}_a = 0. \tag{2.3}$$

They satisfy the following conditions.

(d) g_{ab} is Hermitian with respect to both $K_\pm{}^a{}_b$:

$$K_{\pm ab} + K_{\pm ba} = 0. \tag{2.4}$$

(e) The complex structures $K_\pm{}^a{}_b$ are parallel with respect to the connections $\nabla_{\pm a}$

$$\nabla_{\pm a} K_\pm{}^b{}_c = 0, \tag{2.5}$$

[1] Here and below, indices are raised and lowered by using the metric g_{ab}.

where the connection coefficients $\Gamma_\pm{}^a{}_{bc}$ are given by

$$\Gamma_\pm{}^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} H^a{}_{bc}, \tag{2.6}$$

$\Gamma^a{}_{bc}$ being the usual Levi-Civita connection coefficients.

The connections $\nabla_{\pm a}$ do have a non vanishing torsion $T_{\pm abc}$, which is totally antisymmetric and indeed equal to the 3-form H_{abc} up to sign,

$$T_{\pm abc} = \pm H_{abc}. \tag{2.7}$$

The Riemann tensors $R_{\pm abcd}$ of the $\nabla_{\pm a}$ satisfy a number of relations, the most relevant of which are collected in appendix A.

Usually, in complex geometry, it is convenient to write the relevant tensor identities in the complex coordinates of the underlying complex structure rather than in general coordinates. In biHermitian geometry, one is dealing with two generally non commuting complex structures. One could similarly write the tensor identities in the complex coordinates of either complex structures, but, in this case, the convenience of complex versus general coordinates would be limited. We decided, therefore, to opt for general coordinates throughout the paper. To this end, we define the complex tensors

$$\Lambda_\pm{}^a{}_b = \frac{1}{2}\big(\delta^a{}_b - iK_\pm{}^a{}_b\big). \tag{2.8}$$

The $\Lambda_\pm{}^a{}_b$ satisfy the relations

$$\Lambda_\pm{}^a{}_c \Lambda_\pm{}^c{}_b = \Lambda_\pm{}^a{}_b, \tag{2.9a}$$

$$\Lambda_\pm{}^a{}_b + \overline{\Lambda}_\pm{}^a{}_b = \delta^a{}_b, \tag{2.9b}$$

$$\Lambda_\pm{}^a{}_b = \overline{\Lambda}_{\pm b}{}^a. \tag{2.9c}$$

Thus, the $\Lambda_\pm{}^a{}_b$ are projector valued endomorphisms of the complexified tangent bundle $TM \otimes \mathbb{C}$. The corresponding projection subbundles of $TM \otimes \mathbb{C}$ are the $\pm$ holomorphic tangent bundles $T^{10}_\pm M$.

It turns out that the 3-form H_{abc} is of type $(2,1)+(1,2)$ with respect to both complex structures $K_\pm{}^a{}_b$,

$$H_{def}\Lambda_\pm{}^d{}_a \Lambda_\pm{}^e{}_b \Lambda_\pm{}^f{}_c = 0 \quad \text{and c.c.} \tag{2.10}$$

Other relations of the same type involving the Riemann tensors $R_{\pm abcd}$ are collected in appendix A.

In [9], Gualtieri has shown that biHermitian geometry is related to generalized Kaehler geometry. This, in turn, is part of generalized complex geometry. For a review of generalized complex and Kaehler geometry accessible to physicists, see [11, 10]. Here, we shall restrict ourselves to mention the salient points of these topics.

Let H be a closed 3-form. An H twisted generalized complex structure $\mathcal{J}$ is a section of the endomorphism bundle of $TM \oplus T^*M$ such that $\mathcal{J}^2 = -1$ and $\mathcal{J} = -\mathcal{J}^*$ with respect

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to the canonical inner product of $TM \oplus T^*M$ and $\mathcal{J}$ is integrable with respect to the H twisted Courant brackets of $TM \oplus T^*M$.

There is a pure spinor formulation of generalized complex geometry, which is often very useful. Spinors of the Clifford bundle $C\ell(TM \oplus T^*M)$ are just sections of $\wedge^* T^*M$, i.e. non homogeneous forms. The Clifford action is defined by

$$(X+\xi)\cdot\phi = i_X\phi + \xi\wedge\phi, \tag{2.11}$$

for $X+\xi$ a section of $TM\oplus T^*M$ and ϕ a section of $\wedge^*T^*M$.[2] With each nowhere vanishing spinor ϕ, there is associated the subbundle L_ϕ of $TM\oplus T^*M$ spanned by all sections $X+\xi$ of $TM \oplus T^*M$ such that

$$(X+\xi)\cdot\phi = 0. \tag{2.12}$$

L_ϕ is isotropic. The spinor ϕ is pure if L_ϕ is maximally isotropic. Conversely, with any maximally isotropic subbundle L of $TM \oplus T^*M$, there is associated a nowhere vanishing pure spinor ϕ defined up to pointwise normalization such that $L = L_\phi$. In general, for a given L, ϕ is defined only locally. Thus, L yields a generally non trivial line bundle U_L in $\wedge^*T^*M$, the pure spinor line of L. The above analysis continues to hold upon complexification.

With any H twisted generalized complex structure, there is associated a maximally isotropic subbundle $L_\mathcal{J}$ of $(TM \oplus T^*M)\otimes\mathbb{C}$: $L_\mathcal{J}$ is the $+i$ eigenbundle of $\mathcal{J}$ in $(TM \oplus T^*M)\otimes\mathbb{C}$. In turn, with $L_\mathcal{J}$, there is associated a pure spinor line $U_\mathcal{J}$ defined locally by a pure spinor $\phi_\mathcal{J}$. The integrability of $\mathcal{J}$ is equivalent to

$$d\phi_\mathcal{J} - H\wedge\phi_\mathcal{J} = (X+\xi)\cdot\phi_\mathcal{J}, \tag{2.13}$$

for some section $X+\xi$ of $(TM \oplus T^*M)\otimes\mathbb{C}$ [9].[3]

An H twisted generalized complex structure $\mathcal{J}$ is an H twisted weak generalized Calabi-Yau structure, if the nowhere vanishing pure spinor $\phi_\mathcal{J}$ is globally defined and further

$$d\phi_\mathcal{J} - H\wedge\phi_\mathcal{J} = 0. \tag{2.14}$$

Note that the line bundle U_L is trivial in this case.

If ω is a symplectic structure, then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{2.15}$$

is an untwisted generalized complex structure. Its associated pure spinor is

$$\phi_{\mathcal{J}_\omega} = \exp_\wedge(-i\omega). \tag{2.16}$$

$\phi_{\mathcal{J}_\omega}$ is globally defined and closed. Therefore, $\mathcal{J}_\omega$ is a weak generalized Calabi-Yau structure.

[2]We use the convention $i_X\omega_{a_1\dots a_{p-1}} = X^b\omega_{ba_1\dots a_{p-1}}$ with ω a p-form throughout the paper.

[3]One further has the $\mathrm{Spin}_0(TM \oplus T^*M)$ invariant condition $[\phi_\mathcal{J}\wedge\sigma(\bar\phi_\mathcal{J})]_{\mathrm{top}} \neq 0$, where σ is the automorphism which reverses the order of the wedge product and $[\cdots]_{\mathrm{top}}$ denotes projection on the top form.

If K is a complex structure, then

$$\mathcal{J}_K = \begin{pmatrix} K & 0 \\ 0 & -K^t \end{pmatrix} \tag{2.17}$$

is an untwisted generalized complex structure. Its associated pure spinor is

$$\phi_{\mathcal{J}_K} = \Omega^{(n,0)} \tag{2.18}$$

where $\Omega^{(n,0)}$ is a closed holomorphic volume form. $\phi_{\mathcal{J}_K}$ is only locally defined in general. $\mathcal{J}_K$ is a weak generalized Calabi-Yau structure, if $\Omega^{(n,0)}$ is globally defined. Note that this requires the vanishing of the Chern class $c_1(M)$.

An H twisted generalized Kaehler structure structure consists of a pair of H twisted generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $\mathcal{J}_1, \mathcal{J}_2$ commute and $\mathcal{G} \equiv -\mathcal{J}_1\mathcal{J}_2 > 0$ with respect to the canonical inner product of $TM \oplus T^*M$.

As shown in [9], if $(g, H, K_\pm)$ is a biHermitian structure, then

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} K_+ \pm K_- & (K_+ \mp K_-)g^{-1} \\ g(K_+ \mp K_-) & -({K_+}^t \pm {K_-}^t) \end{pmatrix} \tag{2.19}$$

yield an H-twisted generalized Kaehler structure as defined above.

An ordinary Kaehler structure (g, K) yields simultaneously a symplectic structure $\omega = gK$ and a complex structure K, with which there are associated the generalized complex structures $\mathcal{J}_1 = \mathcal{J}_K$ $\mathcal{J}_2 = \mathcal{J}_\omega$ defined in (2.15), (2.17). Then, $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kaehler structure. If the Kaehler structure (g, K) is Calabi-Yau, then both $\mathcal{J}_1$ $\mathcal{J}_2$ are weak generalized Calabi-Yau structures.

3. The (2,2) supersymmetric sigma model

We shall review next the main properties of the biHermitian (2,2) supersymmetric sigma model, which are relevant in the following analysis.

The base space of the model is a $1+1$ dimensional Minkoskian surface Σ, usually taken to be a cylinder. The target space of the model is a smooth manifold M equipped with a biHermtian structure $(g, H, K_\pm)$. The basic fields of the model are the embedding field x^a of Σ into M and the spinor fields ${\psi_\pm}^a$, which are valued in the vector bundle x^*TM.[4]

The action of biHermitian (2,2) supersymmetric sigma model is given by

$$\begin{aligned} S = \int_\Sigma d^2\sigma \Big[&\frac{1}{2}(g_{ab} + b_{ab})(x)\partial_{++}x^a\partial_{--}x^b \\ &+ \frac{i}{2}g_{ab}(x)({\psi_+}^a\nabla_{+--}{\psi_+}^b + {\psi_-}^a\nabla_{-++}{\psi_-}^b) \\ &+ \frac{1}{4}R_{+abcd}(x){\psi_+}^a{\psi_+}^b{\psi_-}^c{\psi_-}^d\Big], \end{aligned} \tag{3.1}$$

[4]Complying with an established use, here and in the following the indices $\pm$ are employed both to label the two complex structures $K_\pm$ of the relevant biHermitian structure and to denote 2-dimensional spinor indices. It should be clear from the context what they stand for and no confusion should arise.

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where $\partial_{\pm\pm} = \partial_0 \pm \partial_1$,

$$\nabla_{\pm\,\mp\mp} = \partial_{\mp\mp} + \Gamma_{\pm}{}^{\cdot}{}_{c\cdot}(x)\partial_{\mp\mp}x^c \tag{3.2}$$

and the field b_{ab} is related to H_{abc} as

$$H_{abc} = \partial_a b_{bc} + \partial_b b_{ca} + \partial_c b_{ab}. \tag{3.3}$$

The $(2,2)$ supersymmetry variations of the basic fields can be written in several ways. We shall write them in the following convenient form

$$\begin{aligned}
\delta x^a = i\Big[&\alpha^+\Lambda_+{}^a{}_b(x)\psi_+{}^b + \tilde{\alpha}^+\overline{\Lambda}_+{}^a{}_b(x)\psi_+{}^b \qquad (3.4a)\\
&+ \alpha^-\Lambda_-{}^a{}_b(x)\psi_-{}^b + \tilde{\alpha}^-\overline{\Lambda}_-{}^a{}_b(x)\psi_-{}^b\Big],
\end{aligned}$$

$$\begin{aligned}
\delta\psi_\pm{}^a = &-\alpha^\pm\overline{\Lambda}_\pm{}^a{}_b(x)\partial_{\pm\pm}x^b - \tilde{\alpha}^\pm\Lambda_\pm{}^a{}_b(x)\partial_{\pm\pm}x^b \qquad (3.4b)\\
&- i\Gamma_\pm{}^a{}_{bc}(x)\Big[\alpha^+\Lambda_+{}^b{}_d(x)\psi_+{}^d + \tilde{\alpha}^+\overline{\Lambda}_+{}^b{}_d(x)\psi_+{}^d\\
&\qquad + \alpha^-\Lambda_-{}^b{}_d(x)\psi_-{}^d + \tilde{\alpha}^-\overline{\Lambda}_-{}^b{}_d(x)\psi_-{}^d\Big]\psi_\pm{}^c\\
&\pm iH^a{}_{bc}(x)\Big[\alpha^\pm\Lambda_\pm{}^b{}_d(x)\psi_\pm{}^d + \tilde{\alpha}^\pm\overline{\Lambda}_\pm{}^b{}_d(x)\psi_\pm{}^d\Big]\psi_\pm{}^c\\
&\mp \frac{i}{2}\big(\alpha^\pm\Lambda_\pm{}^a{}_d + \tilde{\alpha}^\pm\overline{\Lambda}_\pm{}^a{}_d\big)H^d{}_{bc}(x)\psi_\pm{}^b\psi_\pm{}^c,
\end{aligned}$$

where $\alpha^\pm$, $\tilde{\alpha}^\pm$ are constant Grassmann parameters. δ generates a $(2,2)$ supersymmetry algebra on shell. The action S enjoys $(2,2)$ supersymmetry, so that

$$\delta S = 0. \tag{3.5}$$

The biHermitian $(2,2)$ supersymmetric sigma model is characterized also by two types of R symmetry: the $\mathrm{U}(1)_V$ vector R symmetry

$$\delta_V x^a = 0, \tag{3.6a}$$
$$\delta_V\psi_\pm{}^a = -i\epsilon_V\Lambda_\pm{}^a{}_b(x)\psi_\pm{}^b + i\epsilon_V\overline{\Lambda}_\pm{}^a{}_b(x)\psi_\pm{}^b, \tag{3.6b}$$

and the $\mathrm{U}(1)_A$ axial R symmetry

$$\delta_A x^a = 0, \tag{3.7a}$$
$$\delta_A\psi_\pm{}^a = \mp i\epsilon_A\Lambda_\pm{}^a{}_b(x)\psi_\pm{}^b \pm i\epsilon_A\overline{\Lambda}_\pm{}^a{}_b(x)\psi_\pm{}^b, \tag{3.7b}$$

where ϵ_V, ϵ_A are infinitesimal real parameters. Classically, the action S enjoys both types of R symmetry, so that

$$\delta_V S = \delta_A S = 0. \tag{3.8}$$

As is well known, at the quantum level, the R symmetries are spoiled by anomalies in general. The R symmetry anomalies cancel, provided the following conditions are satisfied [13]:

$$c_1(T_+^{10}M) - c_1(T_-^{10}M) = 0, \qquad \text{vector } R \text{ symmetry}, \tag{3.9a}$$
$$c_1(T_+^{10}M) + c_1(T_-^{10}M) = 0, \qquad \text{axial } R \text{ symmetry}. \tag{3.9b}$$

To generate topological sigma models using twisting, we switch to the Euclidean version of the $(2,2)$ supersymmetric sigma model. Henceforth, Σ is a compact Riemann surface of genus ℓ_Σ. Further, the following formal substitutions are to be implemented

$$\partial_{++} \to \partial_z \tag{3.10a}$$

$$\partial_{--} \to \overline{\partial}_{\overline{z}} \tag{3.10b}$$

$$\psi_+{}^a \to \psi_\theta{}^a \in C^\infty(\Sigma, \kappa_\Sigma{}^{\frac{1}{2}} \otimes x^*TM) \tag{3.10c}$$

$$\psi_-{}^a \to \psi_{\overline{\theta}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma{}^{\frac{1}{2}} \otimes x^*TM) \tag{3.10d}$$

where $\kappa_\Sigma{}^{\frac{1}{2}}$ is any chosen spin structure (a square root of the canonical line bundle κ_Σ of Σ).

The topological twisting of the biHermitian $(2,2)$ supersymmetric sigma model is achieved by shifting the spin of fermions either by $q_V/2$ or $q_A/2$, where q_V, q_A are the fermion's vector and axial R charges, respectively. The resulting topological sigma models will be called biHermitian A and B models, respectively. The twisting can be performed only if the corresponding R symmetry is non anomalous, i.e if the conditions (3.9) are satisfied. The (3.9) can be rephrased as conditions on the the generalized Kaehler structure $(\mathcal{J}_1, \mathcal{J}_2)$ corresponding to the given biHermitian structure $(g, H, K_\pm)$ according to (2.19) [13]. If E_k denotes the $+i$ eigenbundle of $\mathcal{J}_k$ in $(TM \oplus T^*M) \otimes \mathbb{C}$, then the conditions read

$$c_1(E_2) = 0, \qquad A \text{ twist}, \tag{3.11a}$$

$$c_1(E_1) = 0, \qquad B \text{ twist}. \tag{3.11b}$$

R symmetry anomaly cancellation, however, is not sufficient by itself to ensure the consistency of the twisting. Requiring the nilpotence of the BRST charge implies further conditions, namely that

$$d\phi_2 - H \wedge \phi_2 = 0, \qquad A \text{ twist}, \tag{3.12a}$$

$$d\phi_1 - H \wedge \phi_1 = 0, \qquad B \text{ twist}, \tag{3.12b}$$

where the ϕ_k are the globally defined pure spinors associated with the generalized complex structures $\mathcal{J}_k$ [13].

The conditions (3.11), (3.12) are satisfied if the structures $\mathcal{J}_2$, $\mathcal{J}_1$ are twisted weak generalized Calabi-Yau, for the A and B twist, respectively. Further, when this is the case, the BRST cohomology is equivalent to the Lie algebroid cohomology of the relevant generalized complex structure [13]. This remarkable result was one of the achievements of Kapustin's and Li's work.

4. The biHermitian A and B sigma models

As explained in section 3, the biHermitian A and B sigma models are obtained from the biHermitian $(2,2)$ supersymmetric sigma model via a set of formal prescriptions, called A

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and B twist. Concretely, the field content of the biHermitian A sigma model is obtained from that of the $(2,2)$ supersymmetric sigma model via the substitutions

$$\Lambda_+{}^a{}_b(x)\psi_\theta{}^b \to \chi_+{}^a \in C^\infty(\Sigma, x^*T_+^{10}M), \tag{4.1a}$$
$$\overline{\Lambda}_+{}^a{}_b(x)\psi_\theta{}^b \to \overline{\psi}_{+z}{}^a \in C^\infty(\Sigma, \kappa_\Sigma \otimes x^*T_+^{01}M), \tag{4.1b}$$
$$\Lambda_-{}^a{}_b(x)\psi_{\overline{\theta}}{}^b \to \psi_{-\overline{z}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma \otimes x^*T_-^{10}M), \tag{4.1c}$$
$$\overline{\Lambda}_-{}^a{}_b(x)\psi_{\overline{\theta}}{}^b \to \overline{\chi}_-{}^a \in C^\infty(\Sigma, x^*T_-^{01}M). \tag{4.1d}$$

The symmetry variations of the A sigma model fields are obtained from those of the $(2,2)$ supersymmetric sigma model fields (cf. eq. (3.4)), by setting

$$\tilde{\alpha}^+ = \alpha^- = 0, \tag{4.2a}$$
$$\alpha^+ = \tilde{\alpha}^- = \alpha. \tag{4.2b}$$

Similarly, the field content of the biHermitian B sigma model is obtained from that of the $(2,2)$ supersymmetric sigma model via the substitutions

$$\Lambda_+{}^a{}_b(x)\psi_\theta{}^b \to \psi_{+z}{}^a \in C^\infty(\Sigma, \kappa_\Sigma \otimes x^*T_+^{10}M), \tag{4.3a}$$
$$\overline{\Lambda}_+{}^a{}_b(x)\psi_\theta{}^b \to \overline{\chi}_+{}^a \in C^\infty(\Sigma, x^*T_+^{01}M), \tag{4.3b}$$
$$\Lambda_-{}^a{}_b(x)\psi_{\overline{\theta}}{}^b \to \psi_{-\overline{z}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma \otimes x^*T_-^{10}M), \tag{4.3c}$$
$$\overline{\Lambda}_-{}^a{}_b(x)\psi_{\overline{\theta}}{}^b \to \overline{\chi}_-{}^a \in C^\infty(\Sigma, x^*T_-^{01}M). \tag{4.3d}$$

The symmetry variations of the B sigma model fields are obtained from those of the $(2,2)$ supersymmetric sigma model fields, by setting

$$\alpha^+ = \alpha^- = 0, \tag{4.4a}$$
$$\tilde{\alpha}^+ = \tilde{\alpha}^- = \alpha. \tag{4.4b}$$

Inspection of the A, B twist prescriptions reveals that

$$A \text{ twist } \leftrightarrows \; B \text{ twist } \text{ under } \; K_+{}^a{}_b \leftrightarrows -K_+{}^a{}_b. \tag{4.5}$$

The target space geometrical data $(g, H, K_\pm)$, $(g, H, \mp K_\pm)$ have precisely the same properties: they are both biHermitian structures. So, at the classical level, any statement concerning the A (B) model translates automatically into one concerning the B (A) model upon reversing the sign of K_+.[5] For this reason, below, we shall consider only the B twist, unless otherwise stated.

The twisted action S_t is obtained from the $(2,2)$ supersymmetric sigma model action S (3.1) implementing the substitutions (4.3). One finds

$$\begin{aligned} S_t = \int_\Sigma d^2z \bigg[& \frac{1}{2}(g_{ab} + b_{ab})(x)\partial_z x^a \overline{\partial}_{\overline{z}} x^b \\ & + i g_{ab}(x)(\psi_{+z}{}^a \overline{\nabla}_{+\overline{z}} \overline{\chi}_+{}^b + \psi_{-\overline{z}}{}^a \nabla_{-z} \overline{\chi}_-{}^b) \\ & + R_{+abcd}(x) \overline{\chi}_+{}^a \psi_{+z}{}^b \overline{\chi}_-{}^c \psi_{-\overline{z}}{}^d \bigg]. \end{aligned} \tag{4.6}$$

[5] For notational consistency, exchanging $K_+{}^a{}_b \leftrightarrows -K_+{}^a{}_b$ must be accompanied by switching $\alpha^+ \leftrightarrows \tilde{\alpha}^+$.

Similarly the twisted field variations are obtained from the $(2,2)$ supersymmetry field variations (3.4) via (4.3), (4.4). One finds that

$$\delta_t = \alpha s_t, \tag{4.7}$$

where s_t is the fermionic variation operator defined by

$$s_t x^a = i(\overline{\chi}_+{}^a + \overline{\chi}_-{}^a), \tag{4.8a}$$

$$s_t\overline{\chi}_+{}^a = -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c\overline{\chi}_+{}^b, \tag{4.8b}$$

$$s_t\overline{\chi}_-{}^a = -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c\overline{\chi}_-{}^b,$$

$$s_t\psi_{+z}{}^a = -i\Gamma_+{}^a{}_{cb}(x)(\overline{\chi}_+{}^c + \overline{\chi}_-{}^c)\psi_{+z}{}^b - \Lambda_+{}^a{}_b(x)(\partial_z x^b - iH^b{}_{cd}(x)\overline{\chi}_+{}^c\psi_{+z}{}^d), \tag{4.8c}$$

$$s_t\psi_{-\overline{z}}{}^a = -i\Gamma_-{}^a{}_{cb}(x)(\overline{\chi}_+{}^c + \overline{\chi}_-{}^c)\psi_{-\overline{z}}{}^b - \Lambda_-{}^a{}_b(x)(\overline{\partial}_{\overline{z}} x^b + iH^b{}_{cd}(x)\overline{\chi}_-{}^c\psi_{-\overline{z}}{}^d).$$

The action S_t is invariant under s_t,

$$s_t S_t = 0. \tag{4.9}$$

It is straightforward to verify that the ideal of field equations in the algebra of local composite fields is invariant under s_t. One verifies also that

$$s_t{}^2 \approx 0, \tag{4.10}$$

where $\approx$ denotes equality on shell, so that s_t is nilpotent on shell. The proof of these statements is outlined in appendix B. In this way, s_t defines an on shell cohomological complex. s_t corresponds to the BRST charge of the model and its on shell cohomology is isomorphic to the BRST cohomology.

In (4.4), there is no real need for the supersymmetry parameters $\tilde{\alpha}^+$, $\tilde{\alpha}^-$ to take the same value α, since, under twisting both become scalars. If we insist $\tilde{\alpha}^+$, $\tilde{\alpha}^-$ to be independent in (3.4), we obtain a more general symmetry variation

$$\hat{\delta}_t = \tilde{\alpha}^+ s_{t+} + \tilde{\alpha}^- s_{t+} \tag{4.11}$$

where the fermionic variation operators $s_{t\pm}$ are given by

$$s_{t+} x^a = i\overline{\chi}_+{}^a, \tag{4.12a}$$

$$s_{t-} x^a = i\overline{\chi}_-{}^a,$$

$$s_{t+}\overline{\chi}_+{}^a = 0, \tag{4.12b}$$

$$s_{t-}\overline{\chi}_+{}^a = -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c\overline{\chi}_+{}^b,$$

$$s_{t+}\overline{\chi}_-{}^a = -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c\overline{\chi}_-{}^b,$$

$$s_{t-}\overline{\chi}_-{}^a = 0,$$

$$s_{t+}\psi_{+z}{}^a = -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_+{}^c\psi_{+z}{}^b - \Lambda_+{}^a{}_b(x)(\partial_z x^b - iH^b{}_{cd}(x)\overline{\chi}_+{}^c\psi_{+z}{}^d), \tag{4.12c}$$

$$s_{t-}\psi_{+z}{}^a = -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c\psi_{+z}{}^b,$$

$$s_{t+}\psi_{-\overline{z}}{}^a = -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c\psi_{-\overline{z}}{}^b,$$

$$s_{t-}\psi_{-\overline{z}}{}^a = -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_-{}^c\psi_{-\overline{z}}{}^b - \Lambda_-{}^a{}_b(x)(\overline{\partial}_{\overline{z}} x^b + iH^b{}_{cd}(x)\overline{\chi}_-{}^c\psi_{-\overline{z}}{}^d).$$

The action S_t is invariant under both $s_{t\pm}$,

$$s_{t\pm} S_t = 0. \tag{4.13}$$

It is straightforward though lengthy to verify that the ideal of field equations in the algebra of local composite fields is invariant under each $s_{t\pm}$ separately. One can show also that the $s_{t\pm}$ are nilpotent and anticommute on shell

$$s_{t\pm}{}^2 \approx 0, \tag{4.14a}$$

$$s_{t+}s_{t-} + s_{t-}s_{t+} \approx 0. \tag{4.14b}$$

The proof of these relations is outlined again in appendix B. It is easy to verify that s_t and the $s_{t\pm}$ are related as

$$s_t = s_{t+} + s_{t-}. \tag{4.15}$$

Therefore, the $s_{t\pm}$ define an on shell cohomological double complex, whose total differential is s_t, a fact already noticed in [13]. (4.15) corresponds to the decomposition of BRST charge in its left and right chiral components.

The significance of these findings in not clear to us, beyond their ostensible algebraic meaning. As shown in [13], the on shell s_t cohomology, or BRST cohomology, is equivalent to the Lie algebroid cohomology of the H twisted generalized complex structure $\mathcal{J}_1$ underlying the target space biHermitian structure. No interpretation of the double on shell $s_{t\pm}$ cohomology on the same lines is known to us yet.

With each biHermitian sigma model of the form described above, there is associated in canonical fashion a conjugate biHermitian sigma model as follows. If $(g, H, K_\pm)$ is the target space biHermitian structure of the given sigma model, the biHermitian structure (g', H', K') of the conjugate model is given by

$$g'_{ab} = g_{ab}, \tag{4.16a}$$

$$H'_{abc} = -H_{abc}, \tag{4.16b}$$

$$K'_\pm{}^a{}_b = K_\mp{}^a{}_b. \tag{4.16c}$$

The world sheet complex structure of the conjugate model is the conjugate of the world sheet complex structure of the given model. The fields of the conjugate model are related to fields of the given model as

$$x'^a = x^a, \tag{4.17a}$$

$$\overline{\chi}'_+{}^a = \overline{\chi}_-{}^a, \qquad \overline{\chi}'_-{}^a = \overline{\chi}_+{}^a, \tag{4.17b}$$

$$\psi'_{+z'}{}^a = \psi_{-\overline{z}}{}^a, \qquad \psi'_{-\overline{z}'}{}^a = \psi_{+z}{}^a, \tag{4.17c}$$

where $z' = \overline{z}$. It is readily verified that the actions of the given and conjugate model are equal

$$S'_t = S_t. \tag{4.18}$$

Their BRST variations are likewise equal,

$$s'_t = s_t. \tag{4.19}$$

Explicitly, this relations means that for any sigma model field ϕ, $s'_t\phi' = s_t\phi$ upon taking (4.16), (4.17) into account. Similarly, one has

$$s'_{t\pm} = s_{t\mp}. \tag{4.20}$$

The original Kaehler B model studied by Witten in [1, 2] is a particular case of the biHermitian B model expounded here: the Kaehler B model with target space Kaehler structure (g, K) is equal to the biHermitian B model with target space biHermitian structure $(g, H = 0, K_\pm = K)$ up to simple field redefinitions. Similarly, the Kaehler A model with Kaehler structure (g, K) equals the biHermitian A model with biHermitian structure $(g, H = 0, K_\pm = \mp K)$. We note that the fermionic variation s_t of the usual Kaehler B model (eq. (4.2) of [2]) is strictly nilpotent and not simply nilpotent on shell. This property no longer holds in the general biKaehler B model studied in this paper, where s_t is nilpotent only on shell. This follows from eqs. (B.3d), (B.3e) of appendix B. The terms which obstruct the nilpotence of s_t are $\Lambda_+{}^a{}_b\overline{E}_{-z}{}^b$ $\Lambda_-{}^a{}_b\overline{E}_{+\overline{z}}{}^b$, which vanish on shell. For the Kaehler B model, the field $H = 0$ and the complex structures $K_+ = K_-$. This makes these terms vanish for algebraic reasons even off shell.

5. Ghost number and descent

We shall postpone the analysis of the delicate issue whether the biHermitian sigma models described in section 4 are indeed topological field theories to section 6. In this section, we shall study certain properties of the models which are relevant in the computation of topological correlators, namely the ghost number anomaly and the descent formalism. For reasons explained in section 4, we can restrict ourselves to the analysis of the B model.

The biHermitian action S_t, given in eq. (4.6), enjoys, besides the BRST symmetry, the ghost number symmetry, defined by the field variations

$$\delta_{\mathrm{gh}} x^a = 0, \tag{5.1a}$$

$$\begin{aligned} \delta_{\mathrm{gh}}\overline{\chi}_+{}^a &= -i\epsilon_+\overline{\chi}_+{}^a, \\ \delta_{\mathrm{gh}}\overline{\chi}_-{}^a &= -i\epsilon_-\overline{\chi}_-{}^a, \end{aligned} \tag{5.1b}$$

$$\begin{aligned} \delta_{\mathrm{gh}}\psi_{+z}{}^a &= i\epsilon_+\psi_{+z}{}^a, \\ \delta_{\mathrm{gh}}\psi_{-\overline{z}}{}^a &= i\epsilon_-\psi_{-\overline{z}}{}^a, \end{aligned} \tag{5.1c}$$

where $\epsilon_\pm$ are infinitesimal even parameters. Thus,

$$\delta_{\mathrm{gh}} S_t = 0. \tag{5.2}$$

The fields x^a, $\overline{\chi}_+{}^a$, $\overline{\chi}_-{}^a$, $\psi_{+z}{}^a$, $\psi_{-\overline{z}}{}^a$ have ghost number 0, +1, +1, −1, −1, respectively. The fermionic variation operators s_t or $s_{t\pm}$ all carry ghost number +1: their action increases ghost number by one unit.

At the quantum level, the ghost number symmetry is anomalous. Indeed, inspecting the fermionic kinetic terms of the action S_t, through a simple application of the index theorem, it is easy to see that

$$n(\overline{\chi}_+) - n(\psi_{+z}) = \int_\Sigma x^* c_1(T_+^{10}M) + \dim_{\mathbb{C}} M(1-\ell_\Sigma), \tag{5.3a}$$
$$n(\overline{\chi}_-) - n(\psi_{-\overline{z}}) = \int_\Sigma x^* c_1(T_-^{10}M) + \dim_{\mathbb{C}} M(1-\ell_\Sigma), \tag{5.3b}$$

where $n(\overline{\chi}_+)$, $n(\psi_{+z})$ $n(\overline{\chi}_-)$, $n(\psi_{-\overline{z}})$, are the numbers of ${\overline{\chi}_+}^a$, ${\psi_{+z}}^a$, ${\overline{\chi}_-}^a$, ${\psi_{-\overline{z}}}^a$ zero modes, respectively. Generically, $n(\psi_{+z})$, $n(\psi_{-\overline{z}})$ vanish, while $n(\overline{\chi}_+)$, $n(\overline{\chi}_-)$ do not. Consequently, the vacuum carries a non vanishing ghost number charge signaling an anomaly. In quantum correlators, this charge must be soaked up by insertions of fields ${\overline{\chi}_+}^a$, ${\overline{\chi}_-}^a$.

Next, let us consider the field variations corresponding to the symmetry parameters α^+, α^- in (3.4). This means that, in (4.4), we relax the condition $\alpha^+ = \alpha^- = 0$. Upon twisting, α^+, α^- become Grassmann world sheet vector fields α^z, $\alpha^{\overline{z}}$, respectively. Thus, the corresponding fermionic variation operators h_{t+z}, $h_{t-\overline{z}}$ are not scalar: they change the world sheet covariance properties of the fields as indicated by their notation. From (3.4), we obtain easily

$$\begin{aligned}
h_{t+z}x^a &= i{\psi_{+z}}^a, &(5.4a)\\
h_{t-\overline{z}}x^a &= i{\psi_{-\overline{z}}}^a,\\
h_{t+z}{\overline{\chi}_+}^a &= -i{\Gamma_+}^a{}_{bc}(x){\psi_{+z}}^b{\overline{\chi}_+}^c - {\overline{\Lambda}_+}^a{}_b(x)(\partial_z x^b - iH^b{}_{cd}(x){\psi_{+z}}^c{\overline{\chi}_+}^d), &(5.4b)\\
h_{t-\overline{z}}{\overline{\chi}_+}^a &= -i{\Gamma_+}^a{}_{bc}(x){\psi_{-\overline{z}}}^b{\overline{\chi}_+}^c,\\
h_{t+z}{\overline{\chi}_-}^a &= -i{\Gamma_-}^a{}_{bc}(x){\psi_{+z}}^b{\overline{\chi}_-}^c,\\
h_{t-\overline{z}}{\overline{\chi}_-}^a &= -i{\Gamma_-}^a{}_{bc}(x){\psi_{-\overline{z}}}^b{\overline{\chi}_-}^c - {\overline{\Lambda}_-}^a{}_b(x)(\overline{\partial}_{\overline{z}} x^b + iH^b{}_{cd}(x){\psi_{-\overline{z}}}^c{\overline{\chi}_-}^d),\\
h_{t+z}{\psi_{+z}}^a &= -i{\Gamma_-}^a{}_{bc}(x){\psi_{+z}}^b{\psi_{+z}}^c, &(5.4c)\\
h_{t-\overline{z}}{\psi_{+z}}^a &= -i{\Gamma_+}^a{}_{bc}(x){\psi_{-\overline{z}}}^b{\psi_{+z}}^c,\\
h_{t+z}{\psi_{-\overline{z}}}^a &= -i{\Gamma_-}^a{}_{bc}(x){\psi_{+z}}^b{\psi_{-\overline{z}}}^c,\\
h_{t-\overline{z}}{\psi_{-\overline{z}}}^a &= -i{\Gamma_+}^a{}_{bc}(x){\psi_{-\overline{z}}}^b{\psi_{-\overline{z}}}^c.
\end{aligned}$$

The variation operators h_{t+z}, $h_{t-\overline{z}}$ lead to no new symmetry of the action S_t. They would, if the world sheet vector fields α^z, $\alpha^{\overline{z}}$ could be taken (anti)holomorphic, but this is not possible on a generic compact Riemann surface Σ. However, they are useful, as they implement the descent sequence yielding the world sheet 1– and 2-form descendants $\mathcal{O}^{(1)}$, $\mathcal{O}^{(2)}$ of an s_t invariant world sheet 0-form field $\mathcal{O}^{(0)}$ [1, 2]. Let us recall briefly how this works out in detail.

Define the 1-form bosonic variation operators[6]

$$h_{t+} = dz h_{t+z}, \qquad h_{t-} = d\overline{z} h_{t-\overline{z}} \tag{5.5}$$

[6]We assume conventionally that the dz, $d\overline{z}$ anticommute with the fermionic fields ${\overline{\chi}_\pm}^a$, ${\psi_{+z}}^a$, ${\psi_{+\overline{z}}}^a$ and the fermionic variation operartors s_t, $s_{t\pm}$, h_{t+z}, $h_{t-\overline{z}}$.

acting on the algebra of form fields generated by the fields x^a, $\overline{\chi}_+{}^a$, $\overline{\chi}_-{}^a$ and the bosonic world sheet 1-form fields

$$\psi_+{}^a = dz\psi_{+z}{}^a, \qquad \psi_-{}^a = d\overline{z}\psi_{+\overline{z}}{}^a. \tag{5.6}$$

Now, set

$$h_t = h_{t+} + h_{t-}. \tag{5.7}$$

It is straightforward to verify that the ideal of field equations in the algebra of local composite form fields is invariant under h_t and that the on shell relation

$$h_t s_t - s_t h_t \approx -id, \tag{5.8}$$

holds, where $d = dz\partial_z + d\overline{z}\partial_{\overline{z}}$ is the world sheet de Rham differential. The proof of these results is outlined again in appendix B.

Assume now that $\mathcal{O}^{(0)}$ is local 0-form field such that

$$s_t\mathcal{O}^{(0)} \approx 0. \tag{5.9}$$

Define the 1– and 2-form local fields

$$\mathcal{O}^{(1)} = h_t\mathcal{O}^{(0)}, \tag{5.10a}$$

$$\mathcal{O}^{(2)} = \frac{1}{2}h_t\mathcal{O}^{(1)}. \tag{5.10b}$$

Then, from (5.8), (5.9), one has the descent equations

$$s_t\mathcal{O}^{(1)} \approx id\mathcal{O}^{(0)}, \tag{5.11a}$$

$$s_t\mathcal{O}^{(2)} \approx id\mathcal{O}^{(1)}. \tag{5.11b}$$

Consequently, one has

$$s_t\oint_\gamma \mathcal{O}^{(1)} \approx 0, \tag{5.12a}$$

$$s_t\oint_\Sigma \mathcal{O}^{(2)} \approx 0, \tag{5.12b}$$

where γ is a 1-cycle in Σ. In this way, non local BRST invariants can be obtained canonically once a local scalar one is given. These invariants are the operators inserted in topological correlators of the associated topological field theories.

The action of the $h_{t\pm}$ is in fact compatible with the double on shell $s_{t\pm}$ cohomology underlying the on shell s_t cohomology. Indeed the ideal of field equations in the algebra of form fields is separately invariant under the $h_{t\pm}$ and, furthermore, the on shell relations

$$h_{t+}s_{t+} - s_{t+}h_{t+} \approx -i\partial, \tag{5.13a}$$

$$h_{t-}s_{t-} - s_{t-}h_{t-} \approx -i\overline{\partial}, \tag{5.13b}$$

$$h_{t+}s_{t-} - s_{t-}h_{t+} \approx 0, \tag{5.13c}$$

$$h_{t-}s_{t+} - s_{t+}h_{t-} \approx 0 \tag{5.13d}$$

hold, where $\partial = dz\partial_z$ and c.c. are the world sheet Dolbeault operators. One has further on shell relations

$$h_{t\pm}h_{t\pm} \approx 0, \tag{5.14a}$$

$$h_{t+}h_{t-} - h_{t-}h_{t+} \approx 0. \tag{5.14b}$$

See again appendix B for a proof of these relations.

We note that the operators h_t, $h_{t\pm}$ all carry ghost number -1. Under conjugation (cf. eq. (4.16), (4.17)), one has

$$h'_t = h_t \tag{5.15}$$

and

$$h'_{t\pm} = h_{t\mp}. \tag{5.16}$$

6. The biHermitian models are topological

The biHermitian sigma models studied in section 4 should be topological field theories. To check this, one should be able to express the sigma model action as

$$S_t \approx s_t\Psi_t + S_{\mathrm{top}}, \tag{6.1}$$

where $\approx$ denotes on shell equality, Ψ_t is a ghost number -1 topological gauge fermion and S_{top} is a topological action. General arguments indicate that, at the quantum level, when (6.1) holds, the topological sigma model field theory depends generically only on the geometrical data contained in S_{top}, since variations of the geometrical data contained in Ψ_t result in the insertion in topological correlators of BRST cohomologically trivial operators and, so, cannot modify those correlators [1, 2]. For reasons explained in section 4, below we shall restrict ourselves to the analysis of the B model.

In general, the topological action S_{top} is of the form

$$S_{\mathrm{top}} = \int_\Sigma x^*\omega, \tag{6.2}$$

where ω is a 2-form depending on some combinations of the target space geometrical data $(g, H, K_\pm)$. If (6.1), (6.2) hold, the sigma model field theory depends only on those combinations and is independent from the complex structure of the world sheet Σ. If ω is closed,

$$d\omega = 0, \tag{6.3}$$

then S_{top} is invariant under arbitrary infinitesimal variations of x. This condition, however, is not strictly necessary to show the topological nature of the model, though it holds normally.[7] When (6.3) holds, we say that S_{top} is strictly topological.

When $H = 0$, it is straightforward to see that Ψ_t, S_{top} are given by

$$\Psi_t = -\int_\Sigma d^2z\frac{1}{2}g_{ab}(x)\big(\psi_{+z}{}^a\overline{\partial}_{\overline{z}}x^b + \psi_{-\overline{z}}{}^a\partial_z x^b\big), \tag{6.4a}$$

$$S_{\mathrm{top}} = \int_\Sigma d^2z\frac{1}{4}\big(2b_{ab} - iK_{+ab} + iK_{-ab}\big)(x)\partial_z x^a\overline{\partial}_{\overline{z}}x^b. \tag{6.4b}$$

[7] We thank A. Kapustin for pointing this out to us.

The expression of Ψ_t is formally identical to that originally found by Witten in [1, 2]. The action $S_{\rm top}$ is of the form (6.2), (6.3) and so it is indeed strictly topological.

Finding Ψ_t, $S_{\rm top}$ when $H \neq 0$ is far more difficult. In this case, apparently, the target space tensor fields which can be built directly from g, H, $K_\pm$ are not sufficient for constructing a gauge fermion Ψ_t and a topological action $S_{\rm top}$. So far, we have not been able to find the solution of this problem in full generality. We have however found a solution valid in the generic situation, as we illustrate next.

Below, we shall assume that the pure spinor ϕ_1 of the H twisted generalized complex structure $\mathcal{J}_1$ associated with by the biHermitian structure $(g, H, K_\pm)$ via (2.19) can be taken of the form

$$\phi_1 = \exp_\wedge(b + \beta), \tag{6.5}$$

where β is a complex 2-form. In our case, for reasons explained in section 3, $\mathcal{J}_1$ is actually a H twisted weak generalized Calabi-Yau structure and, so, the pure spinor ϕ_1 is globally defined and satisfies (2.14). This requires that β is closed,

$$d\beta = 0. \tag{6.6}$$

Generalized Kaehler structures with the above properties have been considered by Hitchin in [27], where various non trivial examples are worked out in detail.

Now, using (2.19), one verifies that the sections $X + \xi$ of $(TM \oplus T^*M) \otimes \mathbb{C}$ of the form

$$X + \xi = X \mp igK_\pm X, \tag{6.7}$$

with X a section of $T_\pm^{10}M$ are valued in the $+i$ eigenbundle of $\mathcal{J}_1$. Thus, as explained in section 2, these must annihilate the pure spinor ϕ_1 (cf. eq. (2.11)). It is easy to see that this leads to the equation

$$i_X(\beta + b \pm igK_\pm) = 0, \tag{6.8}$$

for any section X of $T_\pm^{10}M$. From here, it follows that there are two 2-forms $\gamma_\pm$ of type $(2, 0)$ with respect to the complex structure $K_\pm$, respectively, such that

$$\beta + b \pm igK_\pm - \overline{\gamma}_\pm = 0. \tag{6.9}$$

This is our basic technical result. We have collected some of the details of the above analysis in appendix C.

The 2-forms $\gamma_\pm$ furnish the hitherto missing elements needed for the construction of the topological gauge fermion Ψ_t and the topological action $S_{\rm top}$. The crucial relations leading to their existence and determining their properties are (6.6), (6.8), which however hinge on the assumption that the pure spinor ϕ_1 is of the form (6.5). There are of course biHermitian structures for which (6.5), is not fulfilled. In general, the pure spinor ϕ_1 is of the form

$$\phi_1 = \exp_\wedge(b + \beta) \wedge \Omega, \tag{6.10}$$

where β is a complex 2-form and Ω is a complex k-form that is decomposable

$$\Omega = \theta_1 \wedge \cdots \wedge \theta_k, \tag{6.11}$$

the θ_i being linearly independent complex 1-forms [8, 9]. The integer k is called type. Demanding that ϕ_1 satisfies the twisted weak Calabi-Yau condition (2.14) entails the equations

$$d\Omega = 0, \tag{6.12a}$$

$$d\beta \wedge \Omega = 0. \tag{6.12b}$$

Requiring further that sections $X + \xi$ of $(TM \oplus T^*M) \otimes \mathbb{C}$ of the form (6.7) annihilate ϕ_1 yields

$$i_X \Omega = 0, \tag{6.13a}$$

$$i_X(\beta + b \pm igK_\pm) \wedge \Omega = 0. \tag{6.13b}$$

for any section X of $T_\pm^{10}M$. In this way, we see that, while (6.6), (6.8) hold when the pure spinor ϕ_1 is of the special form (6.5), they do not necessarily hold when ϕ_1 is of the general form (6.10), though they may. If they do, then 2-forms $\gamma_\pm$ exist and have the same properties as when (6.5) is satisfied.

As shown in refs. [8, 9], the generic even (odd) type twisted generalized complex structures are those of type 0 (1). The twisted generalized complex structures for which (6.5) hold are of type 0. Thus, our analysis covers the generic even type structures and may possibly cover a subset of the remaining structures.

The type k is not necessarily constant and may jump at a locus $C \subset M$ of an even number of units. Type jumping is one of the subtlest aspects of generalized complex geometry [9, 11]. If it does occur, it is possible for the spinor ϕ_1 to have the special form (6.5) at $M \setminus C$, while taking the general form (6.10) at C. In that case, we expect the 2-forms $\gamma_\pm$ to develop some sort of singularity at C. If the embedding field x intersects C, then our analysis below, which assumes the smoothness of the $\gamma_\pm$, may break down. In this way, the locus C may behave as some kind of defect, that is invisible at the classical level, but which may have detectable effects at the quantum level. This however is just a speculation for the time being. At any rate, type jumping occurs only for $\dim_{\mathbb{R}} M \geq 6$. Examples of type jumping from 0 to an higher even value are not easily found.

Under the assumption that the 2-forms $\gamma_\pm$ are available, one can show by explicit computation that (6.1) indeed holds with

$$\Psi_t = -\int_\Sigma d^2z \frac{1}{2}\Big\{\big(g_{ab} + \frac{1}{2}\gamma_{+ab}\big)(x)\psi_{+z}{}^a \overline{\partial}_{\overline{z}} x^b + \big(g_{ab} - \frac{1}{2}\gamma_{-ab}\big)(x)\psi_{-\overline{z}}{}^a \partial_z x^b\Big\}, \tag{6.14a}$$

$$S_{\mathrm{top}} = \int_\Sigma d^2z \frac{1}{4}\big(2b_{ab} - iK_{+ab} + iK_{-ab} - \gamma_{+ab} - \gamma_{-ab}\big)(x)\partial_z x^a \overline{\partial}_{\overline{z}} x^b. \tag{6.14b}$$

The verification requires the use of several non trivial identities involving $\gamma_\pm$ following from (6.6), (6.9), which are conveniently collected in appendix D. From (6.9), it appears that the action S_{top} can be written as

$$S_{\mathrm{top}} = -\int_\Sigma d^2z \frac{1}{2}\overline{\beta}_{ab}(x)\partial_z x^a \overline{\partial}_{\overline{z}} x^b. \tag{6.15}$$

Since β satisfies (6.6), the action $S_{\rm top}$ is again of the form (6.2), (6.3) and, therefore, it is strictly topological. It is quite remarkable that $S_{\rm top}$ is related in simple fashion to the pure spinor ϕ_1 associated with the generalized complex structure $\mathcal{J}_1$.[8]

In the above discussion, we have tacitly assume that the closed 3-form H is exact, so that the 2 form b is globally defined. If H is not exact, b is defined only locally. The combination $\beta + b$ is however globally defined in any case, as ϕ_1 is, and, so, also the 2-forms $\gamma_\pm$ are, by (6.9). If H is not exact, the meaning of the term $\int_\Sigma x^*b$ appearing in the expression of $S_{\rm top}$ must be qualified. If $x(\Sigma)$ is a boundary in the target space M, then $\int_\Sigma x^*b = \int_\Gamma \bar{x}^*H$, where Γ is a 3-fold such that $\partial\Gamma = \Sigma$ and $\bar{x} : \Gamma \to M$ is an embedding such that $\bar{x}|_\Sigma = x$. The value of $\int_\Sigma x^*b$ computed in this way depends on the choice of Γ. In the quantum theory, in order to have a well defined weight $\exp(iS)$ in the path integral for a properly normalized action S, it is necessary to require that $H/2\pi$ has integer periods, so that the cohomology class $[H/2\pi] \in H^3(M,\mathbb{R})$ belongs to the image of $H^3(M,\mathbb{Z})$ in $H^3(M,\mathbb{R})$. If one wants to extend the definition to the general case where $x(\Sigma)$ is a cycle of M, the theory of Cheeger-Simons differential characters is required [28, 29].

We remark that, when $H = 0$, (6.1) holds with Ψ_t, $S_{\rm top}$ given by (6.4a), (6.4b) even if (6.5) does not hold. If it does, however, one can use alternatively (6.14a), (6.14b). Note that (6.14a), (6.14b) do not reduce to (6.4a), (6.4b) when $H = 0$, since the 2-forms $\gamma_\pm$ do not necessarily vanish for $H = 0$. This indicates that the splitting (6.1) of S as a sum of a BRST exact plus a topological term is not unique and can be done in more than one way in general.

Assuming again that the 2-forms $\gamma_\pm$ are available, one has also a chirally split version of (6.1),

$$S_t \approx s_{t+}\Psi_{t+} + s_{t-}\Psi_{t-} + S_{\rm top}, \tag{6.16}$$

where the gauge fermions $\Psi_{t\pm}$ are given by

$$\Psi_{t+} = -\int_\Sigma d^2z \frac{1}{2}\big(g_{ab} + \frac{1}{2}\gamma_{+ab}\big)(x)\psi_{+z}{}^a\overline{\partial}_{\overline{z}}x^b, \tag{6.17a}$$

$$\Psi_{t-} = -\int_\Sigma d^2z \frac{1}{2}\big(g_{ab} - \frac{1}{2}\gamma_{-ab}\big)(x)\psi_{-\overline{z}}{}^a\partial_z x^b, \tag{6.17b}$$

and $S_{\rm top}$ is given by (6.14b). Note also that

$$\Psi_t = \Psi_{t+} + \Psi_{t-} \tag{6.18}$$

When $H = 0$, (6.16) holds in any case with $\gamma_\pm = 0$. The significance of these properties is not clear to us yet.

The results, which we have obtained, albeit still incomplete, shed light on the nature of world sheet and target space geometrical data, on which the quantum field theories associated with the biHermitian A and B sigma models effectively depend. The expressions (6.4b), (6.14b) of $S_{\rm top}$ obtained above show that $S_{\rm top}$ depends only on $\mathcal{J}_1$ (cf. eq. (2.19)). Thus, the quantum biHermitian B model considered here depends effectively

[8]The possibility of a connection of this type was predicted by A. Tomasiello before this analysis was made.

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only on $\mathcal{J}_1$. The quantum biHermitian A model depends instead only on $\mathcal{J}_2$ on account of (4.5). Both models are also evidently independent from the complex structure of the world sheet Σ. These findings confirm earlier results [12, 13].

For the usual Kaehler A model, $H = 0$ and $-K_+ = K_-$. Then, relation (6.1) reduces to the analogous relation originally found by Witten (cf. eqs. (3.3)–(3.5) of [2]). For the Kaehler B model, $H = 0$ and $K_+ = K_-$. Here, however, one has to take into account that Witten calculation holds in strict sense and not simply on shell as the one carried out above. Witten found that (6.1) holds with S_{top} replaced by a functional W of the fermion fields (cf. eq. (4.3)–(4.6) of [2]). However, $W = 0$ on shell as is easy to see. Upon taking this into account, relation (6.1) reproduces Witten's result on shell, as required.

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A. Formulae of biHermitian geometry

In this appendix, we collect a number of useful identities of biHermitian geometry, which are repeatedly used in the calculations illustrated in the main body of the paper. Below $(g, H, K_\pm)$ is a fixed biHermitian structure on an even dimensional manifold M.

1. Relations satisfied by the 3-form H_{abc}.

$$\partial_a H_{bcd} - \partial_b H_{acd} + \partial_c H_{abd} - \partial_d H_{abc} = 0. \tag{A.1}$$

2. Relations satisfied by the connections $\Gamma_\pm{}^a{}_{bc}$.

$$\Gamma_\pm{}^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} H^a{}_{bc}, \tag{A.2a}$$
$$\Gamma_\pm{}^a{}_{bc} = \Gamma_\mp{}^a{}_{cb}, \tag{A.2b}$$

where $\Gamma^a{}_{bc}$ is the Levi-Civita connection of the metric g_{ab}.

3. Relations satisfied by the torsion $T_\pm{}^a{}_{bc}$ of $\Gamma_\pm{}^a{}_{bc}$.

$$T_\pm{}^a{}_{bc} = \pm H^a{}_{bc}, \tag{A.3a}$$
$$T_\pm{}^a{}_{bc} = T_\mp{}^a{}_{cb}. \tag{A.3b}$$

4. Relations satisfied by the Riemann tensor $R_{\pm abcd}$ of $\Gamma_\pm{}^a{}_{bc}$.

$$R_{\pm abcd} = R_{abcd} \pm \frac{1}{2}(\nabla_d H_{abc} - \nabla_c H_{abd}) + \frac{1}{4}(H^e{}_{ad} H_{ebc} - H^e{}_{ac} H_{ebd}), \tag{A.4a}$$
$$R_{\pm abcd} = R_{\mp cdab}, \tag{A.4b}$$

where R_{abcd} is the Riemann tensor of the metric g_{ab}.

Bianchi identities.

$$R_{\pm abcd} + R_{\pm acdb} + R_{\pm adbc} \mp (\nabla_{\pm b} H_{acd} + \nabla_{\pm c} H_{adb} + \nabla_{\pm d} H_{abc}) \\ + H^e{}_{ab} H_{ecd} + H^e{}_{ac} H_{edb} + H^e{}_{ad} H_{ebc} = 0, \tag{A.5a}$$

$$\nabla_{\pm e} R_{\pm abcd} + \nabla_{\pm c} R_{\pm abde} + \nabla_{\pm d} R_{\pm abec} \\ \pm (H^f{}_{ec} R_{\pm abfd} + H^f{}_{cd} R_{\pm abfe} + H^f{}_{de} R_{\pm abfc}) = 0. \tag{A.5b}$$

Other identities

$$R_{\pm abcd} - R_{\pm cbad} = R_{\pm acbd} \pm \nabla_{\pm d} H_{abc}, \tag{A.6a}$$

$$R_{\pm abcd} - R_{\pm cbad} = R_{\mp acbd} \mp \nabla_{\mp b} H_{acd}, \tag{A.6b}$$

$$R_{\pm abcd} - R_{\mp abcd} = \pm \nabla_{\pm d} H_{abc} \mp \nabla_{\pm c} H_{dab} \\ + H^e{}_{ac} H_{ebd} + H^e{}_{da} H_{ebc} - H^e{}_{ab} H_{ecd}. \tag{A.6c}$$

5. The complex structures $K_\pm{}^a{}_c K_\pm{}^c{}_b$.

$$K_\pm{}^a{}_c K_\pm{}^c{}_b = -\delta^a{}_b. \tag{A.7}$$

Integrability

$$K_\pm{}^d{}_a \partial_d K_\pm{}^c{}_b - K_\pm{}^d{}_b \partial_d K_\pm{}^c{}_a - K_\pm{}^c{}_d \partial_a K_\pm{}^d{}_b + K_\pm{}^c{}_d \partial_b K_\pm{}^d{}_a = 0. \tag{A.8}$$

Hermiticity

$$g_{cd} K_\pm{}^c{}_a K_\pm{}^d{}_b = g_{ab}. \tag{A.9}$$

Kaehlerness with torsion

$$\nabla_{\pm a} K_\pm{}^b{}_c = 0. \tag{A.10}$$

6. Other properties.

$$H_{efg} \Lambda_\pm{}^e{}_a \Lambda_\pm{}^f{}_b \Lambda_\pm{}^g{}_c = 0 \quad \text{and c. c.,} \tag{A.11}$$

$$R_{\pm efcd} \Lambda_\pm{}^e{}_a \Lambda_\pm{}^f{}_b = 0 \quad \text{and c. c.,} \tag{A.12}$$

where

$$\Lambda_\pm{}^a{}_b = \frac{1}{2}(\delta^a{}_b - iK_\pm{}^a{}_b) \quad \text{and c. c.} \tag{A.13}$$

B. Some technical calculations

Let $\mathscr{F}$ be the graded commutative algebra of local composite fields generated by the fields x^a, $\overline{\chi}_+{}^a$, $\overline{\chi}_-{}^a$, $\psi_{+z}{}^a$, $\psi_{+\overline{z}}{}^a$. Let $\mathscr{E}$ be the bilateral ideal of $\mathscr{F}$ generated by the composite fields

$$\begin{aligned}
D_{z\bar{z}}{}^a &= -\overline{\nabla}_{+\bar{z}}\partial_z x^a + iR_{+bc}{}^a{}_d(x)\overline{\chi}_+{}^b\psi_{+z}{}^c\overline{\partial}_{\bar{z}}x^d + iR_{-bc}{}^a{}_d(x)\overline{\chi}_-{}^b\psi_{-\bar{z}}{}^c\partial_z x^d \\
&\quad + (\nabla_+{}^a R_{+bcde} + H^{fa}{}_d R_{+bcfe} + H^{fa}{}_e R_{+bcdf})(x)\overline{\chi}_+{}^b\psi_{+z}{}^c\overline{\chi}_-{}^d\psi_{-\bar{z}}{}^e \\
&= -\nabla_{-z}\overline{\partial}_{\bar{z}}x^a + iR_{+bc}{}^a{}_d(x)\overline{\chi}_+{}^b\psi_{+z}{}^c\overline{\partial}_{\bar{z}}x^d + iR_{-bc}{}^a{}_d(x)\overline{\chi}_-{}^b\psi_{-\bar{z}}{}^c\partial_z x^d \\
&\quad + (\nabla_-{}^a R_{-bcde} - H^{fa}{}_d R_{-bcfe} - H^{fa}{}_e R_{-bcdf})(x)\overline{\chi}_-{}^b\psi_{-\bar{z}}{}^c\overline{\chi}_+{}^d\psi_{+z}{}^e, \qquad \text{(B.1a)}\\
\overline{E}_{+\bar{z}}{}^a &= i\overline{\nabla}_{+\bar{z}}\overline{\chi}_+{}^a + R_+{}^a{}_{bcd}(x)\overline{\chi}_+{}^b\overline{\chi}_-{}^c\psi_{-\bar{z}}{}^d, \qquad \text{(B.1b)}\\
\overline{E}_{-z}{}^a &= i\nabla_{-z}\overline{\chi}_-{}^a + R_-{}^a{}_{bcd}(x)\overline{\chi}_-{}^b\overline{\chi}_+{}^c\psi_{+z}{}^d, \\
F_{+\bar{z}z}{}^a &= i\overline{\nabla}_{+\bar{z}}\psi_{+z}{}^a + R_+{}^a{}_{bcd}(x)\psi_{+z}{}^b\overline{\chi}_-{}^c\psi_{-\bar{z}}{}^d, \qquad \text{(B.1c)}\\
F_{-z\bar{z}}{}^a &= i\nabla_{-z}\psi_{-\bar{z}}{}^a + R_-{}^a{}_{bcd}(x)\psi_{-\bar{z}}{}^b\overline{\chi}_+{}^c\psi_{+z}{}^d.
\end{aligned}$$

$\mathscr{E}$ is usually called the ideal of field equations, because the vanishing of its generators (B.1) is equivalent to the imposition of the field equations on the basic fields. The on shell quotient algebra $\mathscr{F}_{\mathscr{E}} = \mathscr{F}/\mathscr{E}$ is thus defined.

The ideal $\mathscr{E}$ is invariant under the fermionic variation operators s_{t+}, s_{t-}, defined in (4.12), as the following calculation shows

$$\begin{aligned}
s_{t+}D_{z\bar{z}}{}^a &= -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c D_{z\bar{z}}{}^b - \nabla_{-z}\overline{E}_{+\bar{z}}{}^a + iR_-{}^a{}_{dbc}(x)\overline{E}_{+\bar{z}}{}^d\overline{\chi}_+{}^b\psi_{+z}{}^c \qquad \text{(B.2a)}\\
&\quad - iR_+{}^a{}_{bcd}(x)\overline{\chi}_+{}^b\overline{E}_{-z}{}^c\psi_{-\bar{z}}{}^d - iR_+{}^a{}_{bcd}(x)\overline{\chi}_+{}^b\overline{\chi}_-{}^c F_{-z\bar{z}}{}^d, \\
s_{t-}D_{z\bar{z}}{}^a &= -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c D_{z\bar{z}}{}^b - \overline{\nabla}_{+\bar{z}}\overline{E}_{-z}{}^a + iR_+{}^a{}_{dbc}(x)\overline{E}_{-z}{}^d\overline{\chi}_-{}^b\psi_{-\bar{z}}{}^c \\
&\quad - iR_-{}^a{}_{bcd}(x)\overline{\chi}_-{}^b\overline{E}_{+\bar{z}}{}^c\psi_{+z}{}^d - iR_-{}^a{}_{bcd}(x)\overline{\chi}_-{}^b\overline{\chi}_+{}^c F_{+\bar{z}z}{}^d, \\
s_{t+}\overline{E}_{+\bar{z}}{}^a &= -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c\overline{E}_{+\bar{z}}{}^b, \qquad \text{(B.2b)}\\
s_{t-}\overline{E}_{+\bar{z}}{}^a &= -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c\overline{E}_{+\bar{z}}{}^b, \\
s_{t+}\overline{E}_{-z}{}^a &= -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c\overline{E}_{-z}{}^b, \qquad \text{(B.2c)}\\
s_{t-}\overline{E}_{-z}{}^a &= -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c\overline{E}_{-z}{}^b, \\
s_{t+}F_{+\bar{z}z}{}^a &= -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_+{}^c F_{+\bar{z}z}{}^b \qquad \text{(B.2d)}\\
&\quad + i\Lambda_+{}^a{}_b(x)[D_{z\bar{z}}{}^b + H^b{}_{cd}(x)\overline{E}_{+\bar{z}}{}^c\psi_{+z}{}^d + H^b{}_{cd}(x)\overline{\chi}_+{}^c F_{+\bar{z}z}{}^d], \\
s_{t-}F_{+\bar{z}z}{}^a &= -i\Gamma_+{}^a{}_{cb}(x)\overline{\chi}_-{}^c F_{+\bar{z}z}{}^b, \\
s_{t+}F_{-z\bar{z}}{}^a &= -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c F_{-z\bar{z}}{}^b, \qquad \text{(B.2e)}\\
s_{t-}F_{-z\bar{z}}{}^a &= -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_-{}^c F_{-z\bar{z}}{}^b \\
&\quad + i\Lambda_-{}^a{}_b(x)[D_{z\bar{z}}{}^b - H^b{}_{cd}(x)\overline{E}_{-z}{}^c\psi_{-\bar{z}}{}^d - H^b{}_{cd}(x)\overline{\chi}_-{}^c F_{-z\bar{z}}{}^d].
\end{aligned}$$

Therefore, s_{t+}, s_{t-} induce fermionic variation operators on the on shell algebra $\mathscr{F}_{\mathscr{E}}$, which we shall denote by the same symbols. The composite variations $s_{t+}{}^2$, $s_{t-}{}^2$, $s_{t+}s_{t-} + s_{t-}s_{t+}$map the field algebra $\mathscr{F}$ into the field equation ideal $\mathscr{E}$, as

$$\begin{aligned}
&s_{t+}{}^2 x^a = 0, \qquad \text{(B.3a)}\\
&s_{t-}{}^2 x^a = 0, \\
&(s_{t+}s_{t-} + s_{t-}s_{t+})x^a = 0, \\
&s_{t+}{}^2\overline{\chi}_+{}^a = 0, \qquad \text{(B.3b)}
\end{aligned}$$

$$s_{t-}{}^2\overline{\chi}_+{}^a = 0,$$
$$(s_{t+}s_{t-} + s_{t-}s_{t+})\overline{\chi}_+{}^a = 0,$$
$$s_{t+}{}^2\overline{\chi}_-{}^a = 0, \tag{B.3c}$$
$$s_{t-}{}^2\overline{\chi}_-{}^a = 0,$$
$$(s_{t+}s_{t-} + s_{t-}s_{t+})\overline{\chi}_-{}^a = 0,$$
$$s_{t+}{}^2\psi_{+z}{}^a = 0, \tag{B.3d}$$
$$s_{t-}{}^2\psi_{+z}{}^a = 0,$$
$$(s_{t+}s_{t-} + s_{t-}s_{t+})\psi_{+z}{}^a = -\Lambda_+{}^a{}_b(x)\overline{E}_{-z}{}^b,$$
$$s_{t+}{}^2\psi_{-\overline{z}}{}^a = 0, \tag{B.3e}$$
$$s_{t-}{}^2\psi_{-\overline{z}}{}^a = 0,$$
$$(s_{t+}s_{t-} + s_{t-}s_{t+})\psi_{-\overline{z}}{}^a = -\Lambda_-{}^a{}_b(x)\overline{E}_{+\overline{z}}{}^b.$$

Therefore, s_{t+}, s_{t-} are nilpotent and anticommute on $\mathscr{F}_{\mathscr{E}}$. This shows (4.14).

Instead of the field algebra $\mathscr{F}$, we consider now the graded commutative form field algebra $\mathscr{F}^\bullet$ generated by the scalar fields x^a, $\overline{\chi}_+{}^a$, $\overline{\chi}_-{}^a$, $\psi_+{}^a$, $\psi_-{}^a$, where $\psi_+{}^a$, $\psi_-{}^a$ are defined in (5.6). Likewise, we consider the bilateral ideal $\mathscr{E}^\bullet$ of $\mathscr{F}^\bullet$ generated by the form field equation fields

$$D^a = dz \wedge d\overline{z} D_{z\overline{z}}{}^a, \tag{B.4a}$$
$$\overline{E}_+{}^a = d\overline{z}\overline{E}_{+\overline{z}}{}^a, \qquad \overline{E}_-{}^a = dz\overline{E}_{-z}{}^a, \tag{B.4b}$$
$$F_+{}^a = d\overline{z} \wedge dz F_{+\overline{z}z}{}^a, \qquad F_-{}^a = dz \wedge d\overline{z} F_{-z\overline{z}}{}^a, \tag{B.4c}$$

where $D_{z\overline{z}}{}^a$, $\overline{E}_{+\overline{z}}{}^a$, $\overline{E}_{-z}{}^a$, $F_{+\overline{z}z}{}^a$, $F_{-z\overline{z}}{}^a$ are defined in (B.1). The on shell form field algebra $\mathscr{F}^\bullet{}_{\mathscr{E}^\bullet} = \mathscr{F}^\bullet/\mathscr{E}^\bullet$ is therefore defined.

The fermionic variation operators s_{t+}, s_{t-} extend in natural and obvious fashion to the field algebras $\mathscr{F}^\bullet$ and $\mathscr{F}^\bullet{}_{\mathscr{E}^\bullet}$, upon assuming conventionally that s_{t+}, s_{t-} anticommute with dz, $d\overline{z}$. In addition to s_{t+}, s_{t-}, we have also the even 1-form variations h_{t+}, h_{t-}, defined by eqs. (5.4), (5.5) and acting on $\mathscr{F}^\bullet$. h_{t+}, h_{t-} preserve $\mathscr{E}^\bullet$, since indeed

$$h_{t+}D^a = 0, \tag{B.5a}$$
$$h_{t-}D^a = 0,$$
$$h_{t+}\overline{E}_+{}^a = -i\Gamma_+{}^a{}_{cb}(x)\psi_+{}^c \wedge \overline{E}_+{}^b \tag{B.5b}$$
$$\qquad + i\overline{\Lambda}_+{}^a{}_b(x)[-D^b + H^b{}_{cd}(x)\psi_+{}^c \wedge \overline{E}_+{}^d + H^b{}_{cd}(x)\overline{\chi}_+{}^c F_+{}^d],$$
$$h_{t-}\overline{E}_+{}^a = 0,$$
$$h_{t+}\overline{E}_-{}^a = 0, \tag{B.5c}$$
$$h_{t-}\overline{E}_-{}^a = -i\Gamma_-{}^a{}_{cb}(x)\psi_-{}^c \wedge \overline{E}_-{}^b$$
$$\qquad + i\overline{\Lambda}_-{}^a{}_b(x)[D^b - H^b{}_{cd}(x)\psi_-{}^c \wedge \overline{E}_-{}^d - H^b{}_{cd}(x)\overline{\chi}_-{}^c F_-{}^d],$$
$$h_{t+}F_+{}^a = 0, \tag{B.5d}$$
$$h_{t-}F_+{}^a = 0,$$
$$h_{t+}F_-{}^a = 0, \tag{B.5e}$$
$$h_{t-}F_-{}^a = 0.$$

We note that the relations $dz \wedge dz = 0$, $d\overline{z} \wedge d\overline{z} = 0$ are crucial for ensuring the validity of the above algebra. Therefore, h_{t+}, h_{t-} induce even 1-form variations on the on shell form field algebra $\mathscr{F}^\bullet{}_{\mathscr{E}^\bullet}$, which we shall denote by the same symbols. An explicit calculation using (4.12), (5.4), (5.5) on the same lines as the above yields the relations

$$\begin{aligned}
&(h_{t+}s_{t+} - s_{t+}h_{t+})x^a = -i\partial x^a, \\
&(h_{t-}s_{t-} - s_{t-}h_{t-})x^a = -i\overline{\partial} x^a, \\
&(h_{t+}s_{t-} - s_{t-}h_{t+})x^a = 0, \\
&(h_{t-}s_{t+} - s_{t+}h_{t-})x^a = 0,
\end{aligned} \tag{B.6a}$$

$$\begin{aligned}
&(h_{t+}s_{t+} - s_{t+}h_{t+})\overline{\chi}_+{}^a = -i\partial\overline{\chi}_+{}^a, \\
&(h_{t-}s_{t-} - s_{t-}h_{t-})\overline{\chi}_+{}^a = -i\overline{\partial}\overline{\chi}_+{}^a + \overline{E}_+{}^a, \\
&(h_{t+}s_{t-} - s_{t-}h_{t+})\overline{\chi}_+{}^a = -\overline{\Lambda}_+{}^a{}_b(x)\overline{E}_-{}^b, \\
&(h_{t-}s_{t+} - s_{t+}h_{t-})\overline{\chi}_+{}^a = 0,
\end{aligned} \tag{B.6b}$$

$$\begin{aligned}
&(h_{t+}s_{t+} - s_{t+}h_{t+})\overline{\chi}_-{}^a = -i\partial\overline{\chi}_-{}^a + \overline{E}_-{}^a, \\
&(h_{t-}s_{t-} - s_{t-}h_{t-})\overline{\chi}_-{}^a = -i\overline{\partial}\overline{\chi}_-{}^a, \\
&(h_{t+}s_{t-} - s_{t-}h_{t+})\overline{\chi}_-{}^a = 0, \\
&(h_{t-}s_{t+} - s_{t+}h_{t-})\overline{\chi}_-{}^a = -\overline{\Lambda}_-{}^a{}_b(x)\overline{E}_+{}^b,
\end{aligned} \tag{B.6c}$$

$$\begin{aligned}
&(h_{t+}s_{t+} - s_{t+}h_{t+})\psi_+{}^a = -i\partial\psi_+{}^a, \\
&(h_{t-}s_{t-} - s_{t-}h_{t-})\psi_+{}^a = -i\overline{\partial}\psi_+{}^a + F_+{}^a, \\
&(h_{t+}s_{t-} - s_{t-}h_{t+})\psi_+{}^a = 0, \\
&(h_{t-}s_{t+} - s_{t+}h_{t-})\psi_+{}^a = \Lambda_+{}^a{}_b(x)F_-{}^b,
\end{aligned} \tag{B.6d}$$

$$\begin{aligned}
&(h_{t+}s_{t+} - s_{t+}h_{t+})\psi_-{}^a = -i\partial\psi_-{}^a, +F_-{}^a \\
&(h_{t-}s_{t-} - s_{t-}h_{t-})\psi_-{}^a = -i\overline{\partial}\psi_-{}^a, \\
&(h_{t+}s_{t-} - s_{t-}h_{t+})\psi_-{}^a = \Lambda_-{}^a{}_b(x)F_+{}^b, \\
&(h_{t-}s_{t+} - s_{t+}h_{t-})\psi_-{}^a = 0,
\end{aligned} \tag{B.6e}$$

and, similarly

$$\begin{aligned}
&h_{t+}h_{t+}x^a = 0, \\
&h_{t-}h_{t-}x^a = 0, \\
&(h_{t+}h_{t-} - h_{t-}h_{t+})x^a = 0,
\end{aligned} \tag{B.7a}$$

$$\begin{aligned}
&h_{t+}h_{t+}\overline{\chi}_+{}^a = 0, \\
&h_{t-}h_{t-}\overline{\chi}_+{}^a = 0, \\
&(h_{t+}h_{t-} - h_{t-}h_{t+})\overline{\chi}_+{}^a = \overline{\Lambda}_+{}^a{}_b(x)F_-{}^b,
\end{aligned} \tag{B.7b}$$

$$\begin{aligned}
&h_{t+}h_{t+}\overline{\chi}_-{}^a = 0, \\
&h_{t-}h_{t-}\overline{\chi}_-{}^a = 0, \\
&(h_{t+}h_{t-} - h_{t-}h_{t+})\overline{\chi}_-{}^a = -\overline{\Lambda}_-{}^a{}_b(x)F_+{}^b,
\end{aligned} \tag{B.7c}$$

$$h_{t+}h_{t+}\psi_+{}^a = 0, \tag{B.7d}$$

$$
\begin{aligned}
&h_{t-}h_{t-}\psi_+{}^a = 0,\\
&(h_{t+}h_{t-} - h_{t-}h_{t+})\psi_+{}^a = 0,\\
&h_{t+}h_{t+}\psi_-{}^a = 0,\\
&h_{t-}h_{t-}\psi_-{}^a = 0,\\
&(h_{t+}h_{t-} - h_{t-}h_{t+})\psi_-{}^a = 0.
\end{aligned} \tag{B.7e}
$$

From these relation (5.13), (5.14) follow immediately.

Similar results hold for the BRST variation s_t and the operator h_t, as is obvious from (4.15) and (5.7), respectively.

C. Type 0 generalized Kaehler structures

We assme first that $H = 0$. Consider the generalized complex $\mathcal{J}_{1/2}$ defined in (2.19). Recalling that $K_\pm{}^t = -gK_\pm g^{-1}$, one finds that

$$
\mathcal{J}_{1/2}\begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}K_+(X+g^{-1}\xi) \pm \frac{1}{2}K_-(X-g^{-1}\xi) \\ \frac{1}{2}gK_+(X+g^{-1}\xi) \mp \frac{1}{2}gK_-(X-g^{-1}\xi), \end{pmatrix} \tag{C.1}
$$

for any section $X+\xi \in C^\infty((TM\oplus T^*M)\otimes\mathbb{C})$. By (C.1), the $+i$ eigenbundle of $\mathcal{J}_1$ contains the sections $X+\xi$ of the form

$$
X+\xi = X \pm gX = X \mp igK_\pm X = X \pm i i_X(gK_\pm), \quad X \in C^\infty(T_\pm^{1,0}M), \tag{C.2}
$$

while the $+i$ eigenbundle of $\mathcal{J}_2$ contains those of the form

$$
\begin{aligned}
X+\xi &= X + gX = X - igK_+X = X + i i_X(gK_+), \quad X \in C^\infty(T_+^{1,0}M),\\
X+\xi &= X - gX = X - igK_-X = X + i i_X(gK_-), \quad X \in C^\infty(T_-^{0,1}M).
\end{aligned} \tag{C.3}
$$

Let ϕ_i be the pure spinor associated to $\mathcal{J}_i$. Then, by (2.11), (2.12),

$$
i_X\phi_i + \xi\wedge\phi_i = 0, \tag{C.4}
$$

for any $X+\xi \in (T\oplus T^*)\otimes\mathbb{C}$ in the $+i$ eigenbundle of $\mathcal{J}_i$, in particular those given in either (C.2) or (C.3). Assume that the pure spinor ϕ_i is of the form

$$
\phi_i = \exp(\beta_i) \tag{C.5}
$$

with β_i a 2-form. Then, (C.2), (C.3), (C.4) imply that

$$
i_X(\beta_1 \pm igK_\pm) = 0, \quad X \in C^\infty(T_\pm^{1,0}M), \tag{C.6}
$$

for $\mathcal{J}_1$, and that

$$
\begin{aligned}
i_X(\beta_2 + igK_+) = 0, \quad X \in C^\infty(T_+^{1,0}M),\\
i_X(\beta_2 + igK_-) = 0, \quad X \in C^\infty(T_-^{0,1}M),
\end{aligned} \tag{C.7}
$$

for $\mathcal{J}_2$. Twisting by $H \neq 0$ simply shifts β_i into $\beta_i + b$. In this way, we recover (6.8) for $\mathcal{J}_1$.

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D. Relevant identities involving $\gamma_\pm$

To begin with, we note that, since $\gamma_\pm$ is a 2-form of type $(2,0)$ with respect to $K_\pm$, one has

$$\overline{\Lambda}_\pm{}^c{}_a\gamma_{\pm cb} = 0. \tag{D.1}$$

This relation will be exploited throughout.

From (6.9), it follows that

$$H \mp id(gK_\pm) - d\gamma_\pm = 0. \tag{D.2}$$

This identity can be cast as

$$\nabla_{\mp a}(\mp iK_{\pm bc} - \gamma_{\pm bc}) = \nabla_{\pm b}\gamma_{\pm ca} + \nabla_{\pm c}\gamma_{\pm ab} \pm H^g{}_{bc}\gamma_{\pm ga} - 2\Lambda_\pm{}^d{}_a H_{dbc}, \tag{D.3}$$

from which one obtains easily

$$\begin{aligned}
&\Lambda_\pm{}^d{}_a\nabla_{\mp d}(\mp iK_{\pm bc} - \gamma_{\pm bc}) && \text{(D.4a)}\\
&\qquad = \Lambda_\pm{}^d{}_a(\nabla_{\pm b}\gamma_{\pm cd} + \nabla_{\pm c}\gamma_{\pm db} \pm H^g{}_{bc}\gamma_{\pm gd} - 2H_{dbc}), &&\\
&\overline{\Lambda}_\pm{}^d{}_a\nabla_{\mp d}(\mp iK_{\pm bc} - \gamma_{\pm bc}) = 0. && \text{(D.4b)}
\end{aligned}$$

From (6.9), it follows also that

$$igK_+ + igK_- + \gamma_+ - \gamma_- = 0. \tag{D.5}$$

From here, one obtains that

$$\overline{\Lambda}_\mp{}^f{}_c\nabla_{\mp a}(\mp iK_{\pm bf} - \gamma_{\pm bf}) = 0. \tag{D.6}$$

Using (D.4), (D.6), it is straightforward to verify that (6.1) holds with Ψ_t, $S_{\rm top}$ given by (6.14).

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